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Extensions to the Kompaneets Equation
and Sunyaev-Zel'dovich Distortion

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ABSTRACT

Analytical expressions are presented for relativistic corrections to the Kompaneets equation and the Sunyaev-Zel'dovich y-distortion. The latter provides a convenient method of inferring both the temperature and Thomson optical depth of the gas in clusters of galaxies using only observations of the spectrum of CMBR photons passing through the cluster. The relativistic correction gives an additional component of the S-Z effect apart from the y-distortion and kinematic S-Z effect. The perturbative method used is shown to provide a very accurate approximations when $kT_{\rm e} \lesssim 100\,{\rm keV}$ but not for higher temperatures.

Subject headings: radiative transfer — scattering — cosmic microwave background — galaxies: clusters: general

1.   Introduction

In the seminal paper, Zel'dovich and Sunyaev (1969), it is shown how interactions of primordial photons with a hot plasma will lead to deviations from a blackbody spectrum of photons. The classical Sunyaev-Zel'dovich (S-Z) effect gives the first order effect on the spectrum as one increases the temperature of the electrons from zero and is not valid as the electron velocities become relativistic. In this *Letter* we give the next order effect, which is important for hot gas in clusters of galaxies, describe a perturbative expansion for computing higher order corrections, and consider the convergence of this expansion.

The classical S-Z distortion was based on the Kompaneets equation (Kompaneets 1957) which describes how the spectrum of photons evolves under the action of Compton scattering off of a stationary plasma of hot electrons. To a first approximation, for non-relativistic electrons ($v^2 \ll c^2$) and soft photons ($\epsilon \ll m_{\rm e}c^2$), kinematics dictates that Compton scattering does not change the photon energy and there should be no change in the photon energy distribution. The Kompaneets equation describes the rate of change in the photon spectrum to order $\frac{v^2}{c^2}$ and $\frac{\epsilon}{m_{\rm e}c^2}$.

The next order term, $\propto \frac{v^4}{c^4}$ or more precisely $\propto \left(\frac{kT_e^2}{m_e c}\right)^2$, was presented in Stebbins 1997a and in this *Letter* we summarize these results while adding other analytical results and applications. Compton scattering has been well studied, by a variety of techniques and non-perturbative computations of the spectral distortion of an incident blackbody spectrum passing through a hot gas at arbitrary temperatures have been published (Fabbri 1981, Rephaeli 1995). The perturbative results of this *Letter* complement the more precise numerical results in that they are analytical and describe the corrections to the classical y-distortion in terms of a given spectral shape. As we shall see, this next order correction provides an good approximation to the numerical results in the temperature regime relevant to the hot gas in clusters of galaxies. Rephaeli and Yankovitch (1997) have already shown that these corrections, combined with relativistic corrections to the X-ray emission can have significant effects on the estimates of the Hubble constant from cluster observations. A more expansive discussion of the perturbative results presented here may be found in Stebbins (1997a,b).

2. Collisional Boltzmann Equation

We describe the state of the primeval gas of photons in terms of the quantum mechanical occupation number in phase space, n_γ. The evolution of n_γ can be described by the collisional Boltzmann equation which has the form

$$\frac{Dn_\gamma(\mathbf{p}_\gamma)}{Dt} = C(\mathbf{p}_\gamma) \tag{1}$$

where $C(\mathbf{p}_\gamma)$ is the scattering term which describes the effect of scattering and $\frac{D}{Dt}$ is a convective derivative along the photon's trajectory in phase space. Assuming isotropy of the photon and electron distribution function, we need only solve for the change in n_γ as a function of the photon energy, ϵ, and not momentum, $\mathbf{p}_\gamma$.[1] The collision term for Compton scattering may be written (Stebbins 1997a,b)

$$C(\epsilon, \Delta) = \int d\beta \, f(\beta) \int d\Delta \left[\frac{1}{(1+\Delta)^3} \overline{S}(\frac{\epsilon}{1+\Delta}, \Delta) \, (1 + n_\gamma(\epsilon)) \, n_\gamma(\frac{\epsilon}{1+\Delta}) \right.$$
$$\left. - \overline{S}(\epsilon, \Delta) \, (1 + n_\gamma(\epsilon(1+\Delta))) \, n_\gamma(\epsilon) \right] . \tag{2}$$

where $\overline{S}(\epsilon, \Delta)$ gives an angle-average of the differential cross-section:

$$\overline{S}(\epsilon, \Delta) = c \, N_e \left\langle (1 - \vec{\beta} \cdot \hat{\mathbf{n}}) \frac{d\sigma}{d\Delta} \right\rangle . \tag{3}$$

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[1]If the photon are not at rest with respect to the electrons then their distribution functions will not both be isotropic in the same rest frame, which will lead to a *direction*-dependent spectral distortion known as the kinematic Sunyaev-Zel'dovich effect, which is a type of Doppler shift.

Here N_e is the electron density, $\hat{\mathbf{n}}$ is the incident electron direction, $\vec{\beta}$ is the electron velocity in units of c, $f(\beta)$ gives the distribution of electron speeds, ϵ is the incident photon energy, and Δ is gives the fractional change in the scattered photon, i.e. $\epsilon' = \epsilon(1 + \Delta)$. The 1st term in the collision integral gives the scattering into the beam and the 2nd term the scattering out of the beam.

3. Fokker-Planck Expansion

As previously mentioned, for non-relativistic electrons and soft-photons, kinematics dictates that the change in the photon energy is small so that the dependence of $\overline{S}(\epsilon, \Delta)$ on it's second argument will be very sharply peaked around $\Delta = 0$. The rest of the dependence on Δ in eq. 2 is much smoother, and may be approximated by it's Taylor series about $\Delta = 0$. This Taylor expansion yield a Fokker-Planck type of approximation. Taylor expanding the Δ-dependence of the distribution function n_γ leaves only derivatives of n_γ evaluated at ϵ. These derivatives may be taken out of the Δ and β integrals and one is left with a differential equation for n_γ rather than an integro-differential equation. The coefficients in the differential equation are determined by the moments of the Δ distribution, which can be determined analytically when the electron distribution is thermal.

The full Klein-Nishina cross-section is a rather complicated function, and for the purposes of the CMBR we are mostly interested in the soft-photon limit. This leads one to Taylor expand the full cross-section in powers of $\alpha = \frac{\epsilon}{m_e c^2}$ and at each order compute eq 3. The 1st term is (Stebbins 1997a,b)

$$\overline{S}(\epsilon, \Delta) = c\, N_e \sigma_T \int_0^1 d\beta\, \beta\, \overline{F}(\Delta, \beta\, \mathrm{sgn}(\Delta)) + \mathcal{O}(\alpha^1) \tag{4}$$

where

$$\overline{F}(\Delta, b) = \mathrm{sgn}(\Delta) \times \mathcal{H}(1 - \frac{(1-b)\,\Delta}{2b}) \times \left[\frac{3(1-b^2)^2(3-b^2)(2+\Delta)}{16b^6} \ln\frac{(1-b)(1+\Delta)}{1+b} \right.$$
$$\left. + \frac{3(1-b^2)(2b-(1-b)\Delta)}{32b^6(1+\Delta)}\left(4(3-3b^2+b^4) + 2(6+b-6b^2-b^3+2b^4)\Delta + (1-b^2)(1+b)\Delta^2\right) \right] \tag{5}$$

where $\mathcal{H}$ is the Lorentz-Heaviside function. With explicit functions such as these it is tedious but straightforward to compute the Fokker-Planck ($\equiv$Taylor series) expansion of eq 2 in powers of α and β or for a thermal electron distribution in powers of $\Theta_e = \frac{kT_e}{m_e c^2}$. Much of the methodology for such expansions was developed in Barbosa (1982).

One may write this Fokker-Planck expansion as

$$\frac{Dn_\gamma}{D\tau} = \frac{1}{\epsilon^2}\frac{\partial}{\partial\epsilon}\epsilon^3\left[\sum_{n\geq 0}\sum_{m\geq 0}\Theta_e^n \alpha^m K^{(n,m)}[n_\gamma]\right] \qquad \Theta_e \equiv \frac{kT_e}{m_e c^2} \qquad \alpha = \frac{\epsilon}{m_e c^2} \tag{6}$$

where $K^{(n,m)}[n_\gamma]$ indicates a, potentially nonlinear, differential operator acting on n_γ which does not depend on $m_e c^2$. The pre-operator, $\frac{1}{\epsilon^2}\frac{\partial}{\partial\epsilon}$, is guaranteed by the form of eq 2 which explicitly conserves photon number. We have used the Thomson optical depth: $\tau = \int dt\, N_e \sigma_T c$ where σ_T is the Thomson cross-section. One finds that to include all terms that contribute to $K^{(n,m)}$ one must Taylor expand to order $2n + m$ in Δ. Since the energy redistribution is a relativistic effect we find that $K^{(0,0)} = 0$. The classical Kompaneets equation contains the two terms:

$$K^{(1,0)}[n_\gamma] = \epsilon \frac{\partial n_\gamma}{\partial \epsilon} \qquad K^{(0,1)}[n_\gamma] = (1 + n_\gamma)\, n_\gamma \ . \tag{7}$$

The 2nd order Δ-expansion of Barbosa (1982) is only accurate enough to infer the one additional term

$$K^{(0,2)}[n_\gamma] = -\frac{63}{5}(1 + n_\gamma)\, n_\gamma \ . \tag{8}$$

The new result (Stebbins97a,b) is

$$K^{(2,0)} = \frac{5}{2}\epsilon \frac{\partial n_\gamma}{\partial \epsilon} + \frac{21}{5}\epsilon^2 \frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \frac{7}{10}\epsilon^3 \frac{\partial^3 n_\gamma}{\partial \epsilon^3} \tag{9}$$

which provides the next order correction for hotter electrons, but not for more energetic photons. Challinor & Lasenby (1997) have confirmed these corrections using an independent derivation and have also computed some additional terms. Combining this new terms with the classical result we find an extended form of the Kompaneets equation:

$$\frac{\partial n_\gamma}{\partial \tau} = \frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\left[\epsilon^3 \left(\frac{kT_e}{m_e c^2}\left(1 + \frac{5}{2}\Theta_e\right)\epsilon \frac{\partial n_\gamma}{\partial \epsilon} + \frac{7}{10}\Theta_e^2\left(6\epsilon^2 \frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \epsilon^3 \frac{\partial^3 n_\gamma}{\partial \epsilon^3}\right) + \alpha\left(1 + n_\gamma\right)n_\gamma\right)\right] \tag{10}$$

which will be more accurate at higher electron temperatures than the classical form.

4. Extended Sunyaev-Zel'dovich Distortion

The idea of the S-Z distortion is that one starts out with a background radiation which is close to a blackbody spectrum, just what we expect to be produced by the early universe, and it is *slightly* distorted by the action of hot ionized gas through the Compton scattering process we have just described. In this small distortion limit we need just substitute a blackbody spectrum with temperature T_γ for n_γ in the right-hand-side of the Kompaneets equation and integrate over τ to obtain the distortion to the spectrum. If we apply this technique to the Fokker-Planck expansion we obtain a series for the total distortion:

$$\Delta n_\gamma = \sum_{n\geq 0}\sum_{m\geq 0} Y_C^{(n,m)}\Delta n_{SZ}^{(n,m)} \qquad Y_C^{(n,m)} = \int d\tau\, \Theta_e^n\, \Theta_\gamma^m \qquad \Theta_\gamma \equiv \frac{kT_\gamma}{m_e c^2} \tag{11}$$

where

$$\Delta n_{SZ}^{(n,m)}(x) = K^{(n,m)}[n_\gamma^{BB}] \qquad n_\gamma^{BB} = \frac{1}{e^x - 1} \qquad x = \frac{\epsilon}{kT_\gamma} \ . \tag{12}$$

From the known $K^{(n,m)}$ one finds

$$\Delta n_{\text{SZ}}^{(0,0)}(x) = 0$$

$$\Delta n_{\text{SZ}}^{(1,0)}(x) = -\Delta n_{\text{SZ}}^{(1,0)}(x) = \frac{5}{63}\Delta n_{\text{SZ}}^{(2,0)}(x) = \frac{xe^x}{(e^x-1)^2}\left(x\frac{e^x+1}{e^x-1} - 4\right)$$

$$\Delta n_{\text{SZ}}^{(2,0)}(x) = \frac{xe^x}{(e^x-1)^2}\left(-10 + \frac{47}{2}x\frac{e^x+1}{e^x-1} - \frac{42}{5}x^2\frac{e^{2x}+4e^x+1}{(e^x-1)^2} + \frac{7}{10}x^3\frac{(e^x+1)(e^{2x}+10e^x+1)}{(e^x-1)^3}\right)$$

$$\tag{13}$$

Since one expects that to each order in energy that a blackbody spectrum is a stationary solution when the electron and photon temperature are equal, one expects the sum rule

$$\sum_{n=0}^{N} \Delta n_{\text{SZ}}^{(n,N-n)}(x) = 0 \ . \tag{14}$$

This sum rule tell us that between the $N+1$ terms in this sum only N give linearly *independent* spectral shapes for the distortion. Applying the sum rule to $N = 2$ one can infer the function $\Delta n_{\text{SZ}}^{(1,1)}(x)$ via the relation $\Delta n_{\text{SZ}}^{(1,1)} = -\Delta n_{\text{SZ}}^{(2,0)} - \Delta n_{\text{SZ}}^{(0,2)}(x)$.

The classical S-Z y-distortion includes only $\Delta n_{\text{SZ}}^{(1,0)}$ and $\Delta n_{\text{SZ}}^{(0,1)}$ and is parameterized by the "y-parameter":

$$y = Y_{\text{C}}^{(1,0)} - Y_{\text{C}}^{(0,1)} = \int d\tau \frac{k(T_{\text{e}} - T_{\gamma})}{m_{\text{e}}c^2} \ . \tag{15}$$

Additional terms in the Fokker-Planck expansion introduces additional parameters such as $Y_{(2,0)}$ which are measures of different moments of the electron or photon temperature.

5. Accuracy and Convergence

Expansions like those of eq.s 6&11 are most useful when good accuracy is obtained with only the first few terms. These expansions give the correct Taylor series expansion in Θ_{e} and Θ_γ about zero, but this may not yield a good approximation even when these variables are small, and in fact these series may not converge for any value of the parameters. As a first check we have compared the series expansion of Δn_γ to order Θ_γ^0 and Θ_{e}^2 to the numerical results of Rephaeli (1995) in fig 1 for $\alpha = 0$ and $kT_{\text{e}} = 5\,\text{keV}$, $10\,\text{keV}$, $15\,\text{keV}$ ($\Theta_{\text{e}} = 0.0098$, 0.0196, and 0.0294). The truncated series yields a good approximation to the numerical results, but with discernible differences. While it is perhaps surprising that such large corrections to the classical S-Z distortion are present at such low temperatures, the fact that the next order correction makes up for most of the total correction indicates that this series will be useful.

To examine the accuracy and convergence of the series expansions one may look at the low frequency limit of the S-Z distortion. By substituting the power series expansion in x of the blackbody spectrum into eq 2 one obtains a power series expansion for the extended S-Z distortion:

$$\Delta n_\gamma^{\text{SZ}}(x) = r_{-1}(\Theta_{\text{e}}) \frac{1}{x} + r_1(\Theta_{\text{e}})\, x + r_3(\Theta_{\text{e}})\, x^3 + \mathcal{O}(x^5, \Theta_\gamma^1) \tag{16}$$
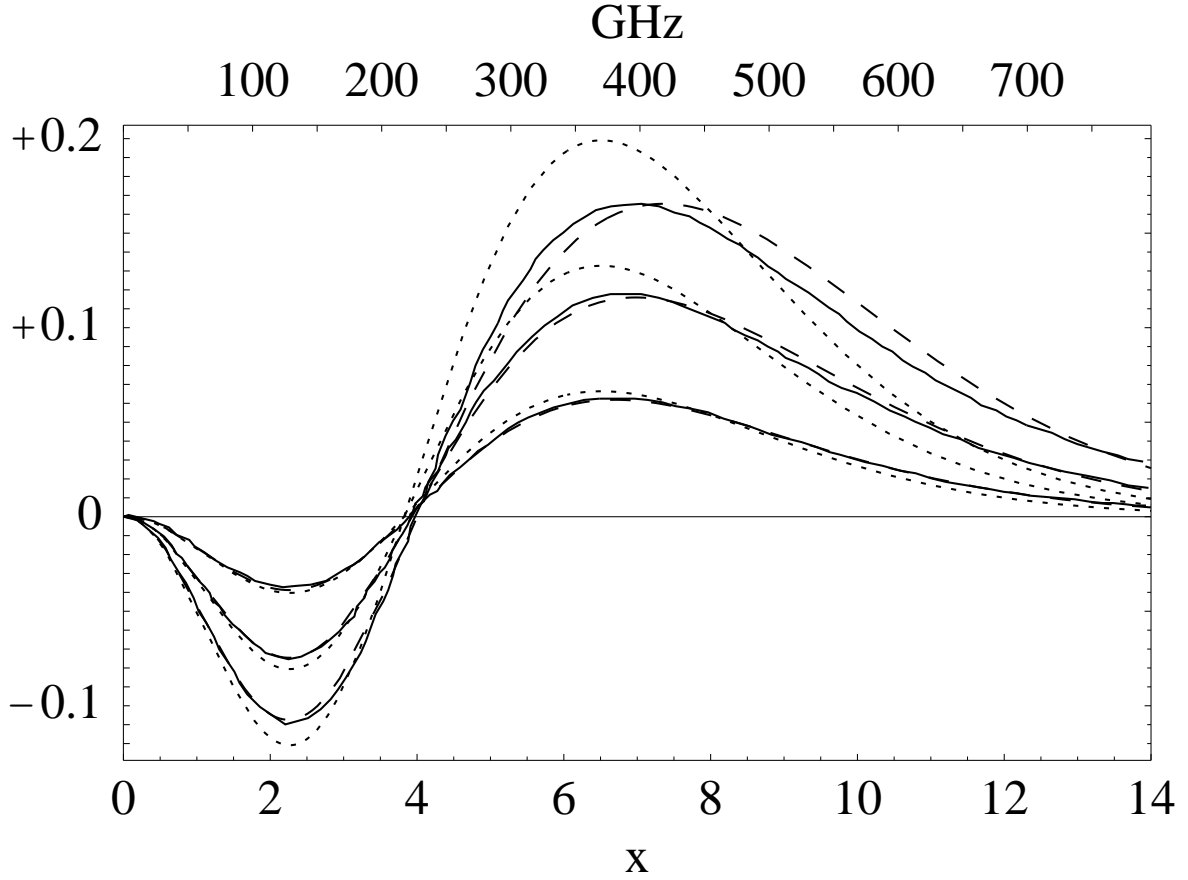
Fig. 1.— Plotted vs. frequency is the shape of the spectral distortion ($x^3 \Delta n(x) \propto$ brightness) to thermal CMBR photons passing through hot gas. Shown are three triplets of curves for gas temperatures of 5, 10, and 15 keV in order of increasing amplitude. In each triplet the dotted curve shows the classical y-distortion, the dashed curve the $\mathcal{O}(\Theta_e^2)$ perturbative prediction, and the solid curve the "true" distortion taken from the numerical results of Rephaeli (1995).

the even powers having zero coefficients. Here we will be interested in the "Rayleigh-Jeans" term, $r_{-1}(\Theta_e)$, which gives the fractional change in brightness at the lowest frequencies. This is given by

$$r_{-1}(\Theta_e) = \int_0^1 d\beta \, \frac{\beta^2 \gamma^5 \, e^{-\gamma/\Theta_e}}{\Theta_e K_2(\frac{1}{\Theta_e})} \left[\frac{9 - 15\,\beta^2 + 10\beta^4}{4\beta^4} - \frac{3\,(3 - 6\beta^2 + 4\beta^4 - \beta^6)}{4\beta^5} \ln\frac{1+\beta}{1-\beta} \right.$$
$$\left. - \frac{3(3 - \beta^2)(1 - \beta^2)^2}{16\beta^6} \left(\ln\frac{1+\beta}{1-\beta} \right)^2 \right] + \mathcal{O}(\Theta_\gamma^1) \tag{17}$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ and K_2 is a modified Bessel function. While one may not be able to express this integral in terms of simple functions it is easily evaluated numerically. One may also compute the Taylor series of $r_{-1}(\Theta_e)$ about zero:

$$r_{-1}(\Theta_e) = -2\Theta_e + \frac{17}{5}\Theta_e^2 - \frac{123}{20}\Theta_e^3 + \frac{1989}{140}\Theta_e^4 - \frac{14403}{320}\Theta_e^5 + \frac{20157}{112}\Theta_e^6 + \mathcal{O}(\Theta_e^7, x^1, \Theta_\gamma^1). \tag{18}$$

which also gives the low frequency limit of the expansion of eq 11 with $m = 0$. The convergence of this series should tell us something about the convergence of eq 11. In fig 2 is plotted the fractional error of truncated Taylor series approximations to r_{-1}. It seems clear that the series does not converge for $kT_e > 80\,\text{keV}$ and likely that it has zero radius of convergence. However below $\sim 100\,\text{keV}$ the 1st few terms in the Taylor series do provide excellent approximations to r_{-1}. If we take this to be representative of the more general series of eq 11 we may conclude that these extensions to the S-Z distortion should be sufficient for application to clusters of galaxies where we expect $kT_e \lesssim 20\text{keV}$. One just need be careful not to expand to such larger orders that the series starts to diverge.[2]

It is not clear whether one might be able to slightly modify eq 11 to obtain a series which is useful at all temperatures. We know that for ultra-relativistic electrons that essentially every scattering will take a photon out of the Rayleigh-Jeans region to much higher energies. Thus we expect the brightness at low frequencies to be degraded to a gray-body spectrum, the brightness decreased by the factor $e^{-\tau}$. This reasoning correctly predicts the limit $r_{-1}(+\infty) = -1$. This asymptotic behaviour makes the error of any approximant which is a polynomial in Θ_e diverge at large Θ_e. Once can hope to improve convergence if one instead Taylor expands in some other variable $X = f(\Theta_e)$ where $f(0) = 0$ and $f(\infty)$ is finite. For example if one takes $X = \Theta_e/(1 + 3\Theta_e)$ the Taylor series of r_{-1} in X,

$$r_{-1}(\Theta_e) = -2X - \frac{13}{5}X^2 - \frac{15}{4}X^3 - \frac{117}{35}X^4 - \frac{3189}{2240}X^5 + \frac{6261}{448}X^6 + \frac{87117}{2560}X^7 + \mathcal{O}(X^{11}) , \tag{19}$$

seems to have good convergence for *all* temperatures. Note that the Taylor series in X contains no more information than that in Θ_e, since one can derive one from the other. What is unclear is whether one can make something like this work for all frequencies.

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[2]These convergence properties are indicative of an *asymptotic* series, an example of which is perturbation theory in quantum field theory.
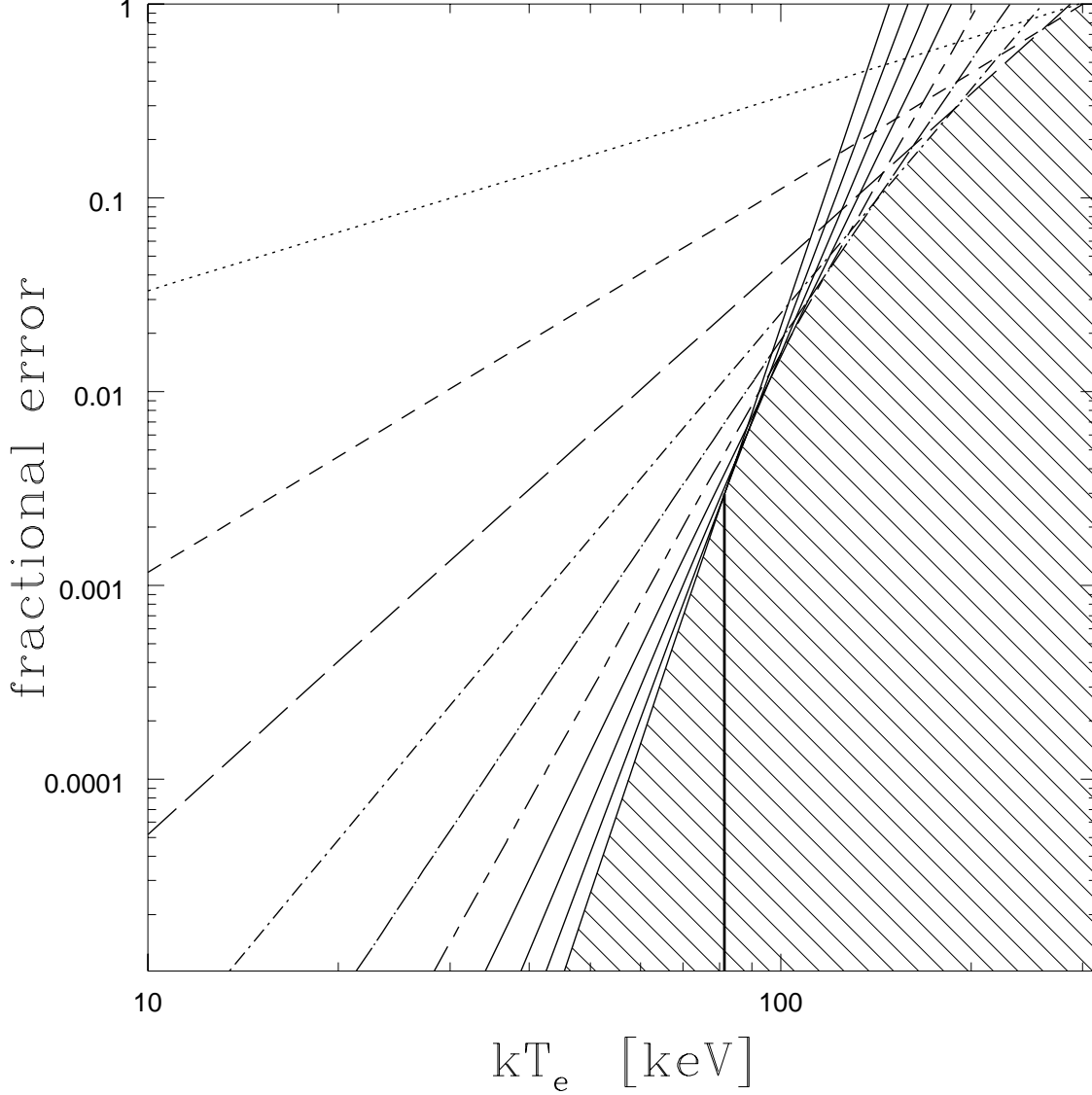
Fig. 2.— Plotted versus electron temperature is the fractional error made by truncated Taylor series approximations to the Rayleigh-Jeans spectral distortion. The curves are: (*dotted*) the classical S-Z effect - a $\mathcal{O}(\Theta_e^1)$ truncation; (*short-dashed*) a $\mathcal{O}(\Theta_e^2)$ truncation; (*long-dashed*) a $\mathcal{O}(\Theta_e^3)$ truncation , and so on up to an $\mathcal{O}(\Theta^{10})$ truncation. The hashed region to the right of the vertical line, and some of the hashed region to its left are accuracies which are not attainable at any order in the Taylor series. Some of the hashed region to the left of the vertical line is attainable by truncating at orders greater than $\mathcal{O}(\Theta_e^{10})$. Note that while the Taylor series is never very accurate when $kT_e > 100\,\text{keV}$, a truncated Taylor series can be extremely accurate at lower temperatures.

At cluster temperatures the accuracy obtained from the 1st few terms in eq 11 is so good that other effects may be the leading source of errors. One is the finite optical depth effect. If one expands in τ as well as in temperature the leading τ^2 term is that given in Fabbri (1981) which is $\sim \mathcal{O}(\tau^2\Theta_e^2)$. For clusters, where $\tau \sim 0.01$, this is liable to be much smaller than the $Y_{(2,0)}\Delta n^{\rm SZ}_{(2,0)}$ correction given above but should be roughly the same order as the next term, $Y_{(3,0)}\Delta n^{\rm SZ}_{(3,0)}$, which we have not computed. To properly compute finite optical depth corrections in a non-uniform medium (such as a cluster) one must really do a proper radiative transfer computation.

6. Application to Clusters

Given that only the first few terms in eq 11 give an excellent approximation to the exact result for cluster temperatures, we may think of the extended S-Z distortion of the CMBR spectrum as consisting a few distinct types of distortions corresponding to the different n and m in that equation. For the CMBR passing through clusters the terms with $m \neq 0$ are negligible and we henceforth ignore them. Of the remaining terms the largest is $\Delta n^{\rm SZ}_{(1,0)}$ (the classical S-Z effect) the 2nd largest is $\Delta n^{\rm SZ}_{(2,0)}$, etc. The amplitude of the classical S-Z distortion is given by $y = Y_{(1,0)}$ and tells one about the product of the temperature and the optical depth. The amplitude of the next term, $Y_{(2,0)}$, can be combined with $Y_{(1,0)}$ to give a weighted measure of the temperature and optical depth separately

$$\overline{kT_e} = m_e c^2 \frac{Y_C^{(2,0)}}{Y_C^{(1,0)}} = \frac{\int d\tau \, (kT_e)^2}{\int d\tau \, kT_e} \qquad \overline{\tau} = \frac{Y_C^{(1,0)\,2}}{Y_C^{(2,0)}} = \frac{[\int d\tau \, (kT_e)]^2}{\int d\tau \, (kT_e)^2} \ . \tag{20}$$

If the electron temperature was constant along a line-of-sight, then we expect each of the ratios, $\frac{Y^{(n+1,0)}}{Y^{(n,0)}}$, to be the same independent of n. However one can use deviations from this rule to detect non-isothermality. For gas as hot as $15\,{\rm keV}$ $\frac{Y_{(2,0)}}{Y_{(1,0)}} \sim 0.03$ but since in the 100-400 GHz region $\frac{\Delta n^{\rm SZ}_{(2,0)}}{\Delta n^{\rm SZ}_{(1,0)}} \sim 5$ the signal from the next order correction to the classical y-distortion is only about 7 times smaller than the classical effect. For many hot clusters one can expect this next order effect to be at the $\frac{\Delta T}{T} \sim 10^{-5}$ level which is certainly accessible with existing telescopes and should also be measurable by the all-sky survey of the PLANCK satellite.

In order to measure quantities like $Y_{(2,0)}$ one must be able to distinguish the $\Delta n^{\rm SZ}_{(2,0)}$ distortion from other effects such as anisotropies, the classical y-distortion, and contamination by dust, synchrotron, and free-free emission. In fig 3 we plot the shape of $\Delta n^{\rm SZ}_{(2,0)}(x)$. One sees that it's shape is quite different than any of the other distortions just mentioned as long as one looks over a sufficient frequency range, say from $100 - 400\,{\rm GHz}$.

Historically the frequency at which the total S-Z distortion to the spectrum is zero has been of great interest. From the eq 11 one may compute the Taylor series describing how this frequency changes as the electron temperature increases. Using $\Delta n^{\rm SZ}_{(1,0)}(x)$ and $\Delta n^{\rm SZ}_{(2,0)}(x)$ one may determine
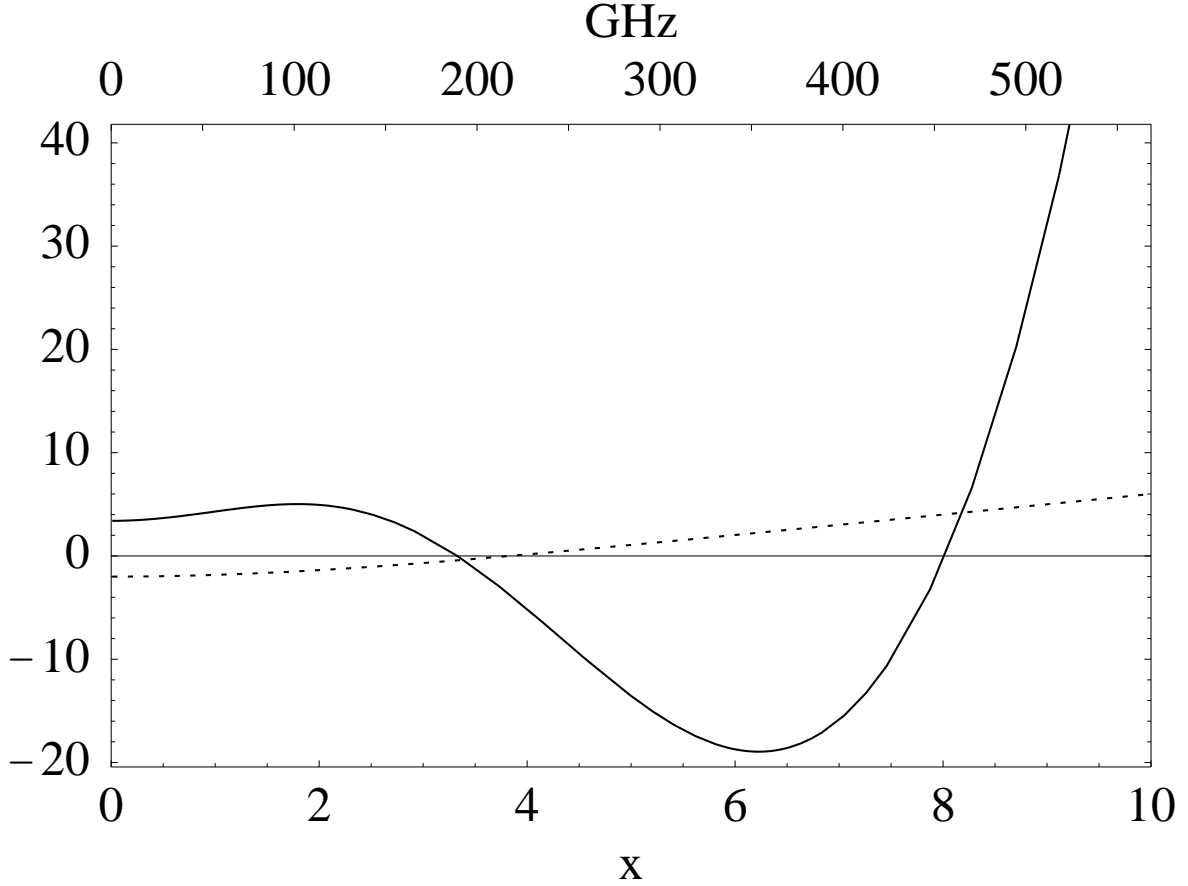
Fig. 3.— Plotted versus frequency is $\frac{\Delta T}{T}$ from the classical y-distortion: $\Delta n^{\rm SZ}_{(1,0)}$ (*dotted line*), and from the next order correction: $\Delta n^{\rm SZ}_{(2,0)}$ (*solid line*). The fact that the solid curve has greater amplitude than the dotted curve indicates that $\frac{Y_{(2,0)}}{Y_{(1,0)}} \sim \frac{kT_{\rm e}}{m_{\rm e}c^2}$ underestimates the size of this correction. The distinct shape of the correction should allow observations with high frequency coverage to separate out this component of the distortion from other sources of foreground contamination.

the linear term,

$$x_0(\Theta_e) = 3.83002 + 4.29189\frac{Y_{(2,0)}}{Y_{(1,0)}} + \mathcal{O}(\Theta_e^2) \tag{21}$$

which agrees with Fabbri (1981). Rephaeli (1995) has pointed out that this linear extrapolation overestimates the shift in x_0 which suggests that the next term in the series is negative and this is indeed found by Challinor & Lasenby (1997).

Traditionally X-ray measurements are used to determine the gas temperature in galaxy clusters. Quantities like those in eq 20 are complementary measures because the X-rays give a temperature weighted by the square of the electron density while the S-Z effect is weighted linearly with the electron density. Thus X-ray temperatures are more weighted toward the cluster centers than S-Z based temperatures. Combining X-ray measurements with S-Z measurements and extensions will allow better empirically based modeling of clusters. More generally one can hope to make temperature determinations using eq 20 in gas clouds which are too diffuse to produce sufficient X-ray luminosity for a good temperature determination. Hubble constant determinations must rely on X-ray measurements since here one makes use of the different density dependencies of X-ray emission and S-Z distortion. One might consider looking for the finite optical depth effects which do depend on a different power of the density, but these are liable to be very difficult to measure (Fabbri 1981).

In summary, it is proposed that one fit for an additional component of the CMBR spectral distortion in the direction of galaxy clusters or other hot gas where a y-distortion has been detected. The amplitude of this additional component, $Y_{(2,0)}$, may be used to gain additional information about the gas temperature and density.

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